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# Geometric interpretation of the Pancharatnam connection and non-cyclic polarization changes

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If the state of polarization of a monochromatic light beam is changed in a cyclical manner, the beam acquires—in addition to the usual dynamic phase—a *geometric phase*. This geometric or *Pancharatnam–Berry phase* equals half the solid angle of the contour traced out on the Poincaré sphere. We show that such a geometric interpretation also exists for the *Pancharatnam connection*, the criterion according to which two beams with different polarization states are said to be in phase. This interpretation offers what is to our knowledge a new and intuitive method to calculate the geometric phase that accompanies non-cyclic polarization changes.

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In 1984 Berry pointed out that a quantum system whose parameters are cyclically altered does not return to its original state but acquires, in addition to the usual dynamic phase, a so-called geometric phase [1]. It was soon realized that such a phase is not just restricted to quantum systems, but also occurs in contexts such as Foucault's pendulum [2]. Also the polarization phenomena described by Pancharatnam [3] represent one of its manifestations. The polarization properties of a monochromatic light beam can be represented by a point on the Poincaré sphere [4]. When, with the help of optical elements such as polarizers and retarders, the state of polarization is made to trace out a closed contour on the sphere, the beam acquires a geometric phase. This *Pancharatnam–Berry phase*, as it is nowadays called, is equal to half the solid angle of the contour subtended at the origin of the sphere [5–10].

In this work we show that such a geometric relation also exists for the so-called *Pancharatnam connection*, the criterion according to which two beams with different polarization states are in phase, i.e., their superposition produces a maximal intensity. This relation can be extended to arbitrary (e.g., non-closed) paths on the Poincaré sphere and allows us to study how the phase builds up for such non-cyclic polarization changes. Our work offers a geometry-based alternative to the algebraic work presented in [11,12].

The state of polarization of a monochromatic beam can be represented as a two-dimensional Jones vector [13] with respect to an orthonormal basis  $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$  as

$$\mathbf{E} = \cos \alpha \hat{\mathbf{e}}_1 + \sin \alpha \exp(i\theta) \hat{\mathbf{e}}_2, \quad (1)$$

with  $0 \leq \alpha \leq \pi/2$ ,  $-\pi \leq \theta \leq \pi$ , and  $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$  ( $i, j = 1, 2$ ). The angle  $\alpha$  is a measure of the relative amplitudes of the two components of the electric vector  $\mathbf{E}$ , and the angle  $\theta$  de-

notes their phase difference. Two different states of polarization,  $A$  and  $B$ , can hence be written as

$$\mathbf{E}_A = (\cos \alpha_A, \sin \alpha_A e^{i\theta_A})^T, \quad (2)$$

$$\mathbf{E}_B = e^{i\gamma_{AB}} (\cos \alpha_B, \sin \alpha_B e^{i\theta_B})^T. \quad (3)$$

Since only relative phase differences are of concern, the overall phase of  $\mathbf{E}_A$  in Eq. (2) is taken to be zero. According to Pancharatnam's connection [5] these two states are in phase when their superposition yields a maximal intensity, i.e., when

$$|\mathbf{E}_A + \mathbf{E}_B|^2 = |\mathbf{E}_A|^2 + |\mathbf{E}_B|^2 + 2 \operatorname{Re}(\mathbf{E}_A \cdot \mathbf{E}_B^*) \quad (4)$$

reaches its greatest value, implying that

$$\operatorname{Im}(\mathbf{E}_A \cdot \mathbf{E}_B^*) = 0, \quad (5)$$

$$\operatorname{Re}(\mathbf{E}_A \cdot \mathbf{E}_B^*) > 0. \quad (6)$$

These two conditions uniquely determine the phase  $\gamma_{AB}$ , except when the states  $A$  and  $B$  are orthogonal.

Let us now consider a sequence of three polarization states with each successive state being in phase with its predecessor. As the initial state we take the basis state  $X$  with Jones vector  $\mathbf{E}_X = (1, 0)^T$ . It follows immediately that any polarization state  $A$  with Jones vector  $\mathbf{E}_A$  as defined by Eq. (2) is in phase with  $X$ . Consider now a third state  $B$ . This state is in phase with  $A$  provided that the angle  $\gamma_{AB}$  in Eq. (3) satisfies relations (5) and (6). Clearly,  $B$  is not in phase with  $X$ , but rather with  $e^{i\gamma_{AB}}X$ . Apparently the total geometric phase that is accrued by following the closed circuit  $XAB$  equals  $\gamma_{AB}$ . This observation allows us to make use of Pancharatnam's classic result which relates the accumulated geometric phase to the solid angle of the geodesic triangle  $XAB$  [3]. According to this result then, the angle (phase)  $\gamma_{AB}$  between the states  $A$  and  $B$

for which they are in phase is given by half the solid angle  $\Omega_{XAB}$  of the triangle  $XAB$  subtended at the center of the Poincaré sphere, i.e.,

$$\gamma_{AB} = \Omega_{XAB}/2. \quad (7)$$

The solid angle  $\Omega_{XAB}$  is taken to be positive (negative) when the circuit  $XAB$  is traversed in a counterclockwise (clockwise) manner. Thus we have  $-2\pi \leq \Omega_{XAB} \leq 2\pi$ , and hence  $-\pi \leq \gamma_{AB} \leq \pi$ . Hence we arrive at the following geometric interpretation of Pancharatnam's connection: The phase  $\gamma_{AB}$  for which the superposition of two beams with polarization states  $A$  and  $B$  yields a maximum intensity equals half the solid angle subtended by their respective Stokes vectors and the Stokes vector corresponding to the basis state  $X$ . We emphasize that  $\gamma_{AB}$  is defined with respect to a certain basis. We return to this point later.

Several consequences follow from the geometric interpretation. First, consider a state  $B$  that lies on the great circle through the points  $A$  and  $X$ . As illustrated in Fig. 1, two cases can be distinguished. If  $B$  is not on the geodesic that connects  $-A$  and  $-X$ , then the curves  $XA$ ,  $AB$ , and  $BX$  cancel each other [see Fig. 1(a)], i.e.,  $\gamma_{AB} = \Omega_{XAB}/2 = 0$ . If  $B$  does lie on the geodesic connecting  $-A$  and  $-X$  [see Fig. 1(b)], then these three curves together constitute the entire great circle and hence  $\gamma_{AB} = \Omega_{XAB}/2 = \pi$ . Consequently, we arrive at

**Corollary 1.** *All polarization states that lie on the great circle that runs through  $A$  and  $X$  and which are not part of the geodesic curve that connects  $-A$  and  $-X$  are in phase with state  $A$ . All other states on the great circle are out of phase with state  $A$ .*

(We exclude the pathological case  $A = \pm X$ .)

The great circle through  $A$  and  $X$  divides the Poincaré sphere into two hemispheres. For all states  $B$  on one

hemisphere, the path  $XAB$  runs clockwise. For  $B$  on the other hemisphere, the path  $XAB$  always runs counterclockwise. Thus we find

**Corollary 2.** *The great circle that runs through  $A$  and  $X$  divides the Poincaré sphere into two halves, one on which all states have a positive phase with respect to  $A$ , and one on which all states have a negative phase with respect to  $A$ .*

Thus far we have not specified the basis vectors in which the Jones vectors are expressed. The two most commonly used are the Cartesian representation and the helicity representation. The Stokes vectors corresponding to the basis state  $X$  are  $(1,0,0)$  and  $(0,0,1)$  in these two bases, respectively. Our results so far are valid for any choice of representation. For computational ease, however, we will from now on make use of the Cartesian basis.

Given two different polarization states  $A$  and  $B$ , we may inquire about the set  $\{B'\}$  of all states which have the same phase difference  $\gamma_{AB}$  with respect to  $A$  as  $B$  has. We begin by noticing that the solid angle  $\Omega_{ABC}$  subtended at the origin of the Poincaré sphere by three unit vectors  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  satisfies the equation [14]

$$\tan\left(\frac{\Omega_{ABC}}{2}\right) = \frac{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})}{1 + \mathbf{B} \cdot \mathbf{C} + \mathbf{A} \cdot \mathbf{C} + \mathbf{A} \cdot \mathbf{B}}. \quad (8)$$

On taking  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  as the Stokes vectors corresponding to states  $A$ ,  $B$ , and  $X$ , i.e.,  $\mathbf{C} = (1,0,0)$ , Eqs. (7) and (8) yield

$$\tan \gamma_{AB} = \frac{A_y B_z - A_z B_y}{1 + B_x + A_x + A_x B_x + A_y B_y + A_z B_z}. \quad (9)$$

For  $\gamma_{AB}$  and  $\mathbf{A}$  fixed, we thus find that the three components of  $\mathbf{B}$  must satisfy the relation

$$c_x B_x + c_y B_y + c_z B_z + D = 0, \quad (10)$$

with the coefficients  $c_x$ ,  $c_y$ ,  $c_z$ , and  $D$  given by

$$c_x = \tan \gamma_{AB}(1 + A_x), \quad (11)$$

$$c_y = \tan \gamma_{AB} A_y + A_z, \quad (12)$$

$$c_z = \tan \gamma_{AB} A_z - A_y, \quad (13)$$

$$D = c_x. \quad (14)$$

The solutions of Eq. (10) form a plane. In addition, the vector  $\mathbf{B}$  must be of unit length, ensuring that it lies on the Poincaré sphere. The intersection of the plane and the sphere is a circle that runs through  $B$ . Finding two other points on this circle defines it uniquely. It can be verified by substitution that the Stokes vectors  $-\mathbf{A}$  and  $-\mathbf{X}$  both satisfy Eq. (10). Hence, for all states on the circle that runs through  $B$ ,  $-\mathbf{A}$ , and  $-\mathbf{X}$ , the phase  $\gamma_{AB}$  has the same value, mod  $\pi$ . Since the plane defined by Eq. (10) does, in general, not include the origin of the Poincaré sphere, this circle is not a great circle. This is illustrated in Fig. 2, where the circle through  $B$  is drawn as dashed. The dashed circle intersects the great circle through  $A$  and  $X$  at the points  $-\mathbf{A}$  and  $-\mathbf{X}$ . According to Corollary 2,  $\gamma_{AB}$  changes sign at these points. Since Eq. (9) defines the

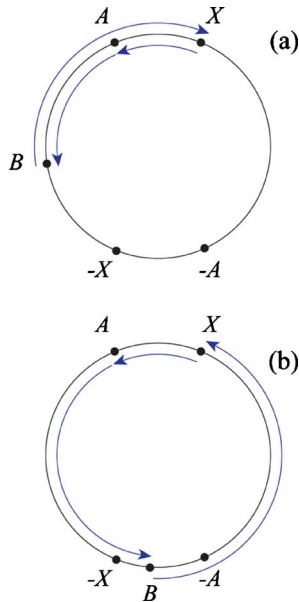


Fig. 1. (Color online) The great circle through  $A$ ,  $B$ , and basis state  $X$ . (a) If state  $B$  does not lie on the segment between  $-A$  and  $-X$ , then the sum of the three geodesics  $XA$ ,  $AB$ , and  $BX$  is zero. (b) If  $B$  lies on the segment between  $-A$  and  $-X$ , then the sum of the three geodesics equals the great circle.

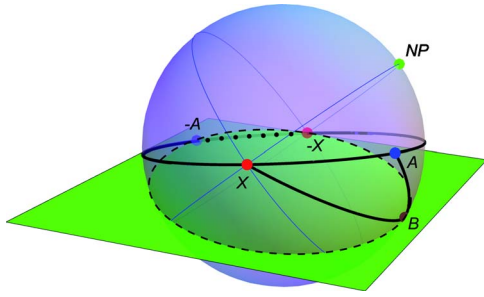


Fig. 2. (Color online) Illustration of the intersection of the plane given by Eq. (10) and the Poincaré sphere. This intersection is a circle (indicated by the dashed curve) that runs through the points  $-A$ ,  $-X$ , and  $B$ . All points on the circle segment that runs from  $-A$  to  $B$  to  $-X$  constitute the set  $\{B'\}$  of states that have the same phase difference  $\gamma_{AB}$  with respect to  $A$  as the state  $B$ . The great circle through  $A$  and  $X$  is shown as a solid-dotted curve.

phase modulo  $\pi$ , it follows that  $\gamma_{AB}$  undergoes a  $\pi$  phase jump at these points. We thus arrive at

**Corollary 3.** Consider the circle through  $-A$ ,  $-X$ , and  $B$ . It consists of two segments, both with end points  $-A$  and  $-X$ . The segment which includes  $B$  equals the set  $\{B'\}$  of states such that  $\gamma_{AB'} = \gamma_{AB}$ . The other segment represents states for which  $\gamma_{AB'} = \gamma_{AB} \pm \pi$ .

It can be shown that the plane-sphere intersection is always a circle, and not just a single point, if the pathological case  $A = \pm X$  is excluded. If, for a fixed state  $A$ , the state  $B$  is being varied, the plane given by Eq. (10) rotates along the line connecting  $-A$  and  $-X$ .

We now demonstrate how our geometric interpretation implies that for a fixed state  $A$  the phase  $\gamma_{AB}$  may vary in different ways when the state  $B$  is moved across the Poincaré sphere. We specify the position of  $B$  by spherical coordinates  $(\phi, \theta)$ , where  $0 \leq \phi \leq 2\pi$  and  $0 \leq \theta \leq \pi$  represent the azimuthal angle and the angle of inclination, respectively. If  $A$  is taken to be at the south pole and  $B = B(\phi)$  lies on the equator, then

$$\gamma_{AB} = \frac{\Omega_{XAB}}{2} = \frac{1}{2} \int_{\pi/2}^{\pi} \int_0^{\phi} \sin \theta d\phi' d\theta = \frac{1}{2} \phi. \quad (15)$$

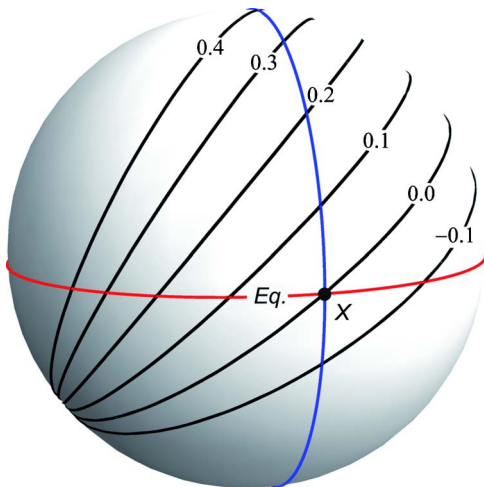


Fig. 3. (Color online) Selected contours of the phase  $\gamma_{AB}$  for the case  $A = (0, 0.8, 0.6)$ . The basis state  $X$ , the equator (*Eq.*), and the meridian through  $X$  are also shown.

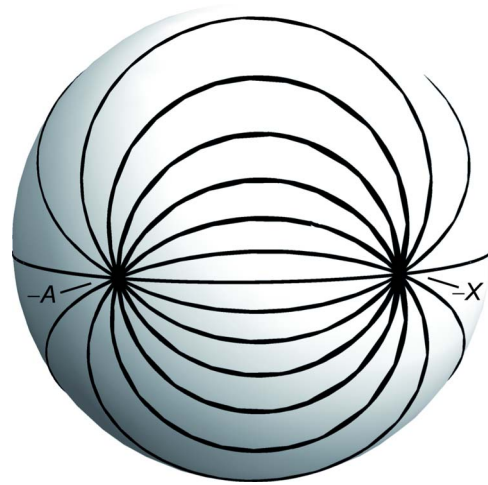


Fig. 4. (Color online) Contours of equal phase of  $\gamma_{AB}$  for the case that the state  $A$  is taken to be  $(0.6, 0, 0.8)$ . Two singular points with opposite topological charges can be seen at  $-A$  and  $-X$ .

Clearly, the phase varies linearly with the angle  $\phi$  in this case.

Let us now consider the contours of equal phase  $\gamma_{AB}$  as shown in Fig. 3. It is seen that the intersections of the contours with the equator are not equidistant. Hence in this case the phase depends in a nonlinear way on the angle  $\phi$ .

The singular behavior, finally, of the phase is a direct consequence of the fact that two anti-podal states  $A$  and  $-A$  do not interfere with each other [see the remark below Eq. (6)]. From Eq. (8) it follows that the phase is antisymmetric under the interchange of the points  $C = X$  and  $A$ . Hence we expect two singular points, namely,  $-A$  and  $-X$ , with opposite topological charges ( $\pm 1$ ). This is illustrated in Figs. 4 and 5. We note that the existence of singular points is in agreement with the “hairy ball theorem” due to Brouwer [15], according to which there is no non-vanishing continuous tangent vector field on a sphere in  $\mathbb{R}^3$ . This implies that  $\nabla \gamma_{AB}$  has at least one zero, in this case at the two singularities.

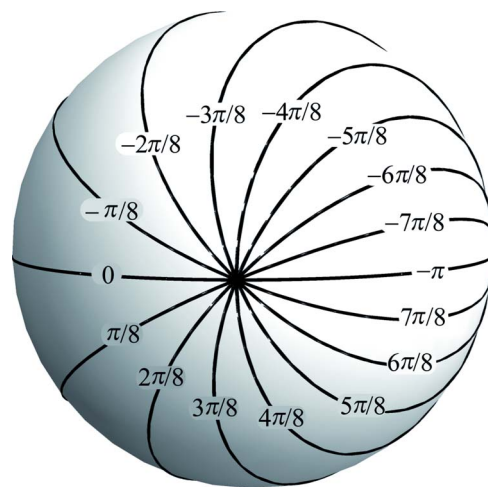


Fig. 5. (Color online) Contours of equal phase of  $\gamma_{AB}$  for the case that the state  $A$  is taken to be  $(0, 0, 1)$ . The singularity at  $-A$  is seen to have a topological charge of  $+1$ .



Let us now apply our results for the Pancharatnam connection to study the geometric phase for an arbitrary, i.e., non-closed, path  $ABC$  on the Poincaré sphere. The successive states are assumed to be in phase. Therefore the geometric phase accumulated on this path equals

$$\gamma_{ABC} \equiv \gamma_{AB} + \gamma_{BC} = (\Omega_{XAB} + \Omega_{XBC})/2 = \Omega_{XABC}/2, \quad (16)$$

where  $\Omega_{XABC}$  is the *generalized solid angle* of the path  $X \rightarrow A \rightarrow B \rightarrow C \rightarrow X$ .  $\Omega_{XABC}$  can consist of two triangles (see Fig. 6), whose contribution is positive or negative depending on their handedness.

Now we keep states  $A$  and  $C$  fixed and study how the geometric phase  $\gamma_{ABC}$  changes when state  $B$  is varied. We will show that this change, in contrast to  $\gamma_{AB}$ , is independent of the choice of basis vectors. Consider the phase  $\gamma'_{ABC}$  in a non-Cartesian basis (for example, the helicity basis) whose first basis state we call  $N$ . We then have, in analogy to Eq. (16),

$$\gamma'_{ABC} \equiv \gamma'_{AB} + \gamma'_{BC} = (\Omega_{NAB} + \Omega_{NBC})/2 = \Omega_{NABC}/2. \quad (17)$$

Also,

$$\Omega_{NABC} - \Omega_{XABC} = \Omega_{NABC} + \Omega_{CBAX} = \Omega_{NAXC}. \quad (18)$$

The justification of the last step of Eq. (18) is illustrated in Fig. 7. It follows on using Eqs. (16)–(18) that

$$\gamma'_{ABC} - \gamma_{ABC} = \Omega_{NAXC}/2. \quad (19)$$

The term  $\Omega_{NAXC}/2$  is a constant, independent of  $B$ , i.e., the geometric phase in both representations differs by a constant only. Hence the variation of the geometric phase with  $B$  is independent of the choice of the basis, as it should be for an observable quantity. This is in contrast to  $\gamma_{AB}$ , which explicitly depends on the choice of basis, as is evident from Eqs. (2) and (3).

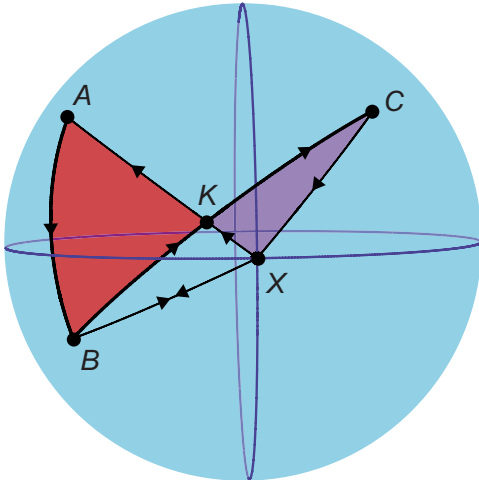


Fig. 6. (Color online) Illustration of the generalized solid angle  $\Omega_{XABC}$ . In going from state  $A$  to state  $B$ , the beam acquires a geometric phase equal to half the solid angle  $\Omega_{XAB}$ , which is positive. In going from  $B$  to  $C$  the acquired phase equals half the solid angle  $\Omega_{XBC}$ , which is negative. Since the triangle  $BKX$  does not contribute, this is equivalent to the generalized solid angle  $\Omega_{XABC}$ , which equals half the solid angle of the triangle  $ABK$  (positive), plus half the solid angle of the triangle  $XKC$  (negative).

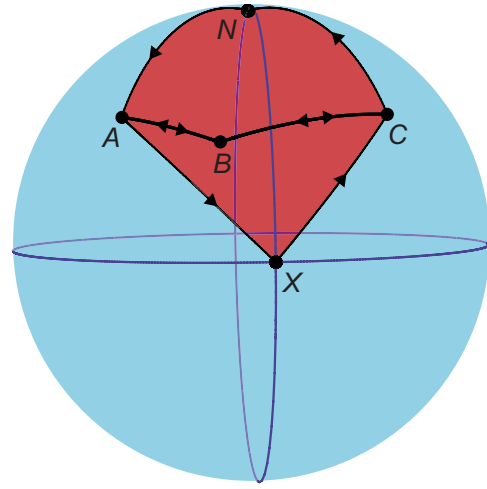


Fig. 7. (Color online) Illustration of the equality  $\Omega_{NABC} + \Omega_{CBAX} = \Omega_{NAXC}$ . Such a construction can be made for any choice of states.

The behavior of  $\gamma_{ABC}$  on varying  $B$  can be linear [16], nonlinear [17], or singular [18–20], as we have also shown for  $\gamma_{AB}$ . However  $\gamma_{AB}$  has singularities at  $B = -A$  and  $B = -X$ . The first is due to the orthogonality of  $A$  and  $-A$ , while the second is a consequence of the choice of representation. The phase  $\gamma_{ABC}$  is singular only at  $B = -A$  and  $B = -C$ , and not at  $B = -X$ .

In conclusion, we have shown how the Pancharatnam connection may be interpreted geometrically. Our work offers a geometry-based approach to calculate the Pancharatnam–Berry phase associated with non-cyclic polarization changes. As such it is an alternative to the algebraic treatments presented in [11,12]. Our approach can be extended to the description of geometric phases in quantum mechanical systems.

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